

Lecture 23 - Buoyancy, Fluids in motion

Buoyancy

Archimedes' principle: The buoyancy force (B) acting on a body is the weight of fluid which that body displaces.

For example let's calculate the buoyancy force of the human body in air. The volume of the human body is roughly $V \approx 2 * 0.2 * 0.5 = 0.2m^3$. The density of air at sea level is approximately $\rho_{air} = 1.25kg/m^3$, so that the buoyancy for on the human body in air is about $B = \rho Vg \approx 2.45N$. This means that when we weigh ourselves on the scales, the weight that the scales measure is actually a little bit less than our true weight. The error is $\delta w \approx 0.25kg$ or about 1/2 lb. This is an example of the effect of a fluid on the measured weight of an object. We shall look at that more closely below. There are two general cases of an object in a fluid: If it is submerged, or if it is floating.

Submerged object

If an object of volume mass M and volume V is submerged in a fluid, then the net force on the object is given by,

$$F_{net} = Mg - B = \rho_{fluid}Vg - Mg = (\rho_{fluid} - \rho_{object})Vg \quad (1)$$

If the density of the object $\rho_{object} = M/V$ is greater than the density of fluid ρ_{fluid} , then the force is negative and the object sinks, while if the density of the object is lower than that of the fluid, we have to impose a force on it to keep it submerged.

Lets consider the case where the density of the object is greater than the density of the fluid in which it is submerged. Another way of thinking about this is illustrated in the calculation we did above for the human body in air. That is, we can weight of the object is lower in the fluid than in vaccuum, and the observed weight is just F_{net}/g . For example, consider weighing a piece of gold of volume V in air and then weigh the same piece of gold in water. Since gold has a density of $19.3 * 10^3kg/m^3$ (so that it has a specific gravity of 19.3), when we weigh it in air, the scales read $w_{in\ air} = (\rho_g - \rho_a)Vg$, while if we weigh it in water, $w_{in\ water} = (\rho_g - \rho_w)Vg$. Note also that if we take the ratio of these two quantities, we get,

$$\frac{w_{in\ air}}{w_{in\ water}} = \frac{\rho_g - \rho_a}{\rho_g - \rho_w} = \frac{19.3 - .00125}{19.3 - 1} \approx \frac{19.3}{18.3} \quad (2)$$

This equation may be solved for ρ_g . In general if we have a material of unknown density, we can find its density by doing a measurement in two fluids of known density and using an equation of this sort.

Floating objects

If an object has a density less than the fluid in which it is placed, then it floats (e.g. ice in water). In that case, the volume of water displaced has a mass equal to the mass of the object, so that,

$$Mg = \rho_{object}Vg = \rho_{fluid}V_{submerged}g \quad (3)$$

where $V_{submerged}$ is the volume of the object which is under the surface of the fluid. We then have,

$$\frac{\rho_{object}}{\rho_{fluid}} = \frac{V_{submerged}}{V_{object}} \quad (4)$$

The right hand side is the fraction of the volume of the object which is submerged. Lets do the calculation for an iceberg. In that case we have $\rho_{object} = \rho_{ice} = 920kg/m^3$ (note that $10^3kg/m^3 = 1g/cc = 1g/ml$). We thus find that for an iceberg floating in water,

$$\frac{V_{submerged}}{V_{iceberg}} = 0.92 \quad (5)$$

which means that 92 percent of an iceberg is below the surface of the water. For that reason sailing a ship through waters containing icebergs is extremely dangerous unless the ship's hull can withstand an impact with one.

Fluids in motion

I. Continuity equation(mass conservation): What goes in comes out!

We have all been in rivers where there are some parts which flow swiftly and others where there are deep quiet pools. How does that work?? Clearly if no water leaks out of the river and there is no additional water entering the river, the water that goes in has to come out. The simple statement that in a fluid flow, what goes in has to come out is a "conservation law". In the case of water we can say the the mass flow is conserved. The mass of water which flows across a dam spillway is given by, $A\rho v$ (which has units of mass/time), where v is the velocity of the flow. In general a flow is something per unit time, for example current is charge per unit time, power is energy per unit

time etc. As remarked above, what flows into the dam has to flow out of the dam, across the spillway. In fact if we measure the flow across any surface crossing the river we must have the same mass flow, which may be written as,

$$A_1\rho_1v_1 = A_2\rho_2v_2. \quad (6)$$

This equation is called a continuity equation and it ensures that mass flow is conserved. Water and many other dense fluids are nearly incompressible, so that $\rho_1 = \rho_2$, so that the continuity equation reduces to $A_1v_1 = A_2v_2$. Physically this means that narrow constrictions in the river have to have fast flow and wide open parts of the river have slow flow.

Bernoulli's (1700-1782) equation: Conservation of Work/Energy

Consider a pipe through which an incompressible fluid is flowing, in a steady flow (no turbulence). Consider a segment of the fluid, at two times t and t' . During the time interval $\Delta t = t' - t$, a small volume of fluid V is moved along the pipe. At one end of the fluid segment, the pipe has area A_1 , the fluid is flowing with velocity v_1 , the pressure is P_1 and the fluid segment moves a distance Δx_1 during time interval Δt . At the other end of the segment of fluid there is a second set of values of these quantities, $A_2, v_2, P_2, \Delta x_2$. Since the fluid is incompressible, the density is a constant and we have $A_1\Delta x_1 = A_2\Delta x_2 = V$.

Now consider the work/energy theorem applied to the fluid segment. The pressure P_1 does work to push the fluid segment through the pipe, while the pressure P_2 resists the flow, the external work is then,

$$W_e = F_1\Delta x_1 - F_2\Delta x_2 = P_1A_1\Delta x_1 - P_2A_2\Delta x_2 = (P_1 - P_2)V \quad (7)$$

The change in kinetic energy is,

$$\Delta KE = \frac{1}{2}m_2v_2^2 - \frac{1}{2}m_1v_1^2 = \frac{1}{2}\rho V(v_2^2 - v_1^2) \quad (8)$$

The change in gravitational potential energy is,

$$\Delta PE = m_2gy_2 - m_1gy_1 = \rho Vg(y_2 - y_1) \quad (9)$$

Using the work-energy theorem, ie $W_e = \Delta KE + \Delta PE + E_{dis}$, and assuming that the dissipation is zero, we find that,

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho gy_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho gy_2 \quad (10)$$

This is Bernoulli's equation, which is often written as,

$$P + \frac{1}{2}\rho v^2 + \rho gy = \text{constant} \quad (11)$$

As an example consider airflow around an airplane wing. The airflow over the top part of the wing has speed $v_2 = 245\text{m/s}$, while the airflow over the bottom of the wing has speed $v_1 = 222\text{m/s}$. The density of air is approximately $\rho = 1.29\text{kg/m}^3$. Since the wing is quite thin, the change in gravitational PE is negligible, so we have,

$$P_2 - P_1 = -\frac{1}{2} * 1.29 * (245^2 - 222^2) = 6.93 * 10^3 \text{Pa} \quad (12)$$

Recall that atmospheric pressure at sea level is, on average, $1.01 * 10^5 \text{Pa}$, so that the flow induced pressure difference is quite a surprisingly large. The fact that it is negative indicates that the pressure is LOWER on the high velocity side of the wing. Airplane airfoils are designed to produce lift, so the high velocity side of the flow is on the upper side of the wing. Racecars on the other hand want to grip the road so their airfoils are designed to have the faster airflow on the lower side of the airfoil. We can estimate the amount of mass that the pressure above can lift through $F = P * \text{Area}$. The larger the wing area, the larger the lift. A Boeing 747-400 has a wing area of $A = 541\text{m}^2$, so that the lift force at these airspeeds and at this air density is $F = 3,740,130\text{N}$. This force can lift a mass of $F/g = 382,174\text{kg}$. The mass of the plane itself is roughly $150,000\text{kg}$ empty, and it carries $63,500$ gallons of fuel on long flights, eg to Australia. The altitude to which a plane can fly is limited by the air density. The air density at $30,000\text{ft} \approx 10,000\text{m}$ is about $1/3$ that at sea level, so the lift is also reduced by this amount. Note that high velocity implies low pressure, IN the fluid. This is counterintuitive as we know that if a high velocity fluid hits us (e.g. a water canon), then the pressure on us is large. We have to carefully distinguish between the pressure in the fluid as predicted by Bernoulli and the pressure on a target which is completely different.